Lemma: X = sch., TFAE

(1) $\forall x \in X$, $\mathcal{O}_{X,x}$ is a normal domain;

(2) $\forall$ affine open $\mathrm{Spec}(A) \subseteq X$, A is normal ($A_\mathfrak{p}$ is a normal domain $\forall \mathfrak{p} \subseteq A$ prime).

(3). $\exists$ open cover $X = \bigcup_i \mathrm{Spec}(A_i)$ w/ A_i normal.

Defn. Such schemes are called <u>normal schemes</u>.

Lemma: normal $\Rightarrow$ reduced.

CA Facts: (1) R = reduced ring having finitely minimal primes. TFAE.

(a) R is normal

(b) R is int'lly closed in its total ring of fractions $Q(R)$ ($S^{-1}R$ for $S \subseteq R$ non-zerodiv.)

(c) $R = \prod_{i=1}^{m} R_i$, R_i normal domain.

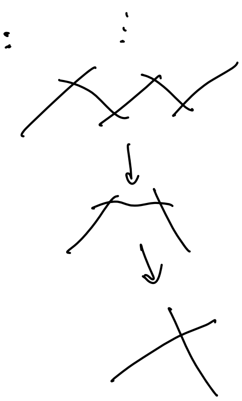
key pt in pf: $Q(R) = R_{\mathfrak{p}_1} \times R_{\mathfrak{p}_2} \times \cdots \times R_{\mathfrak{p}_n}$.

(2). $X = \mathrm{Spec}(A)$, normal + conn'd $\not\Rightarrow$ integral!

(1) $A_\infty := \operatorname{colim}_{i \in \mathbb{N}} A_i$ has no idempotent.

$A_m \subseteq k[x]^{2^m+1}$.

Example:



$A_2 \subseteq k[x]^5$. $(f_0, f_0(1), f_1, f_1(1), f_2)$

$\uparrow$

$A_1 \subseteq k[x] \times k[x] \times k[x]$

$= \{(f_0, f_1, f_2) \mid f_i(1) = f_{i+1}(0)\}$ $(f_0, f_0(1), f_1)$

$\uparrow$

$A_0 \subseteq k[x] \times k[x]$

$= \{(f_0, f_1) \mid f_0(1) = f_1(0)\}$. (f_0, f_1)

(3) $(x-1, 0) \cdot (0, x) = (0,0)$.

(2) but $\forall \mathfrak{p} \subseteq A_\infty$; $\mathfrak{p}_n := \mathfrak{p} \cap A_n$. If both $\mathfrak{p}_n$ & $\mathfrak{p}_{n+1}$ are nodes, the transition map $(A_n)_{\mathfrak{p}_n} \longrightarrow (A_{n+1})_{\mathfrak{p}_{n+1}}$ factors.

$$(A_n)_{\mathfrak{p}_n} \searrow k[x]_{(x)} \nearrow (A_{n+1})_{\mathfrak{p}_{n+1}}$$

(min'l primes of A_∞ have localizations k).

Point Set Top. Fact: X = top. space. Irred. components of $X :=$ max'l irred. subset of X.

(1) $T \subseteq X$ irred $\Rightarrow$ $\overline{T} \subseteq X$ irred.

(2) Irred. components are closed.

(3) $X = \bigcup_{Z \in \{\text{Irred. comp of } X\}} Z$.

Lemma: X = scheme, any irred. comp. is of the form: $\mathrm{Spec}(A) \overset{\text{open}}{\subseteq} X$, $\mathfrak{p} \subseteq A$ minimal prime, $Z = \overline{\{\mathfrak{p}\}}$ closure in X.

Prop. X loc. Noeth. normal $\Rightarrow$ $X = \coprod_{\lambda \in \Lambda} X_\lambda$, where each $X_\lambda \subseteq X$ is an irred. comp. & open & normal.

| X Noeth. $\Rightarrow$ $X = \coprod_{i=1}^{n} X_i$ ...

pf: pick $T \subseteq X$, irred. comp. $\forall$ affine open $\underbrace{\mathrm{Spec}(A)}_{=: \mathcal{U}} \subseteq X$,
$\underset{\overline{\{t\}}}{\shortparallel}$

If $t \in \mathrm{Spec}(A)$, then t must be a min. prime,

A Noeth. normal $\Rightarrow$ $A = \prod A_i$, and $t \leftrightarrow A_1$.

$\Rightarrow$ $T \cap \mathrm{Spec}(A) = \overline{\{t\}}^{\mathcal{U}}$ is an open in $\mathrm{Spec}(A)$ $\square$.

Lemma: X int'l normal $\Rightarrow$ $\underbrace{\Gamma(X, \mathcal{O}_X)}_{=: R}$ normal domain.

pf: R is a domain by def'n. If $f = \frac{a}{b} \in \mathrm{Frac}(R)$, int'l over R.

Say $f^n + \sum_{i=0}^{n-1} a_i f^i = 0$ for $a_i \in R$.

Then $\forall$ affine open $\emptyset \neq \mathcal{U} = \mathrm{Spec}(A) \subseteq X$, $b|_{\mathcal{U}} \neq 0$ (otherwise $V(b) \cup \mathcal{U}^c = X$ $\Rightarrow V(b) = X$ ※)

and $\frac{a|_{\mathcal{U}}}{b|_{\mathcal{U}}}$ satisfies monic eqn $/A$. $\Rightarrow$ $\frac{a|_{\mathcal{U}}}{b|_{\mathcal{U}}} =: f_{\mathcal{U}} \in A$.

Clearly $\forall$ affine opens $V \subseteq \mathcal{U}$, $f_{\mathcal{U}}|_V = f_V$. (by looking at $\mathrm{Frac}(\mathcal{O}(\mathcal{U})) \cong \mathrm{Frac}(\mathcal{O}(V))$)
glues to f as $b \cdot (\text{glue outcome}) = a$ (it's so on every $\mathcal{U}$).